Majority-vote on directed Small-World networks

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Abstract: On *directed* Small-World networks the Majority-vote model with noise is now studied through Monte Carlo simulations. In this model, the order-disorder phase transition of the order parameter is well defined in this system. We calculate the value of the critical noise parameter q_c for several values of rewiring probability p of the directed Small-World network. The critical exponentes β/ν, γ/ν and $1/\nu$ were calculated for several values of p.

Keywords: Monte Carlo simulation, vote , networks, nonequilibrium.

Introduction

Sumour and Shabat [1, 2] investigated Ising models on *directed* Barabási-Albert networks [3] with the usual Glauber dynamics. No spontaneous magnetisation was found, in contrast to the case of *undirected* Barabási-Albert networks [4, 5, 6] where a spontaneous magnetisation was found lower a critical temperature which increases logarithmically with system size. Lima and Stauffer [7] simulated directed square, cubic and hypercubic lattices in two to five dimensions with heat bath dynamics in order to separate the network effects form the effects of directedness. They also compared different spin flip algorithms, including cluster flips [8], for Ising-Barabási-Albert networks. They found a freezing-in of the magnetisation similar to [1, 2], following an Arrhenius law at least in low dimensions. This lack of a spontaneous magnetisation (in the usual sense) is consistent with the fact that if on a directed lattice a spin S_j influences spin S_i, then spin S_i in turn does not influence S_j, and there may be no well-defined total energy. Thus, they show that for the same scale-free networks, different algorithms give different results. It has been argued that nonequilibrium stochastic spin systems on regular square lattice with up-down symmetry fall in the universality class of the equilibrium Ising model [9]. This conjecture was found in several models that do not obey detailed balance [10, 11, 12, 13, 14]. Lima [15, 16] investigated the majority-vote model on *directed* and *undirected* Barabási-Albert

network and calculated the β/ν, γ/ν, and $1/\nu$ exponents and these are different from the Ising model and depend on the values of connectivity z of the Barabási-Albert network. Campos *et al.* [17] investigated the majority-vote model on *undirected* small-world network by rewiring the two-dimensional square lattice. These small-world networks, aside from presenting quenched disorder, also possess long-range interactions. They found that the critical exponents γ/ν and β/ν are different from the Ising model and depend on the rewiring probability. However, it was not evident whether the exponent change was due to the disordered nature of the network or due to the presence of long-range interactions. Lima *et al.* [18] studied the majority-vote model on Voronoi-Delaunay random lattices with periodic boundary conditions. These lattices posses natural quenched disorder in their conecctions. They showed that presence of quenched connectivity disorder is enough to alter the exponents β/ν and γ/ν the pure model and therefore that is a relevant term to such non-equilibrium phase-transition. Now, we calculate the same β/ν, γ/ν, and $1/\nu$ exponents for majority-vote model on *directed* small-world networks of Sánchez et al. [19], and for these networks the exponents are different from the two-dimensional Ising model and independent on the values of rewiring probability p of the *directed* small-world networks. Here we study the cases for $p > 0$, but we verify that when $p = 0$ our results are agree with results of J. M. Oliveira [13].

Model and Simulaton

We consider the majority-vote model, on *directed* small-world networks, defined [13, 20, 18, 21] by a set of "voters" or spins variables σ taking the values $+1$ or -1, situated on every site of an *directed* small-world networks with $N = L \times L$ sites, were L is the side of square lattice, and evolving in time by single spin-flip like dynamics with a probability w_i given by

$$w_i(\sigma) = \frac{1}{2}\Big[1 - (1 - 2q)\sigma_i S\Big(\sum_{\delta=1}^{k_i} \sigma_{i+\delta}\Big)\Big],$$ (1)

where $S(x)$ is the sign ± 1 of x if $x \neq 0$, $S(x) = 0$ if $x = 0$, and the sum runs over four neighbours of σ_i. In this network, created for Sánchez et al. [19],
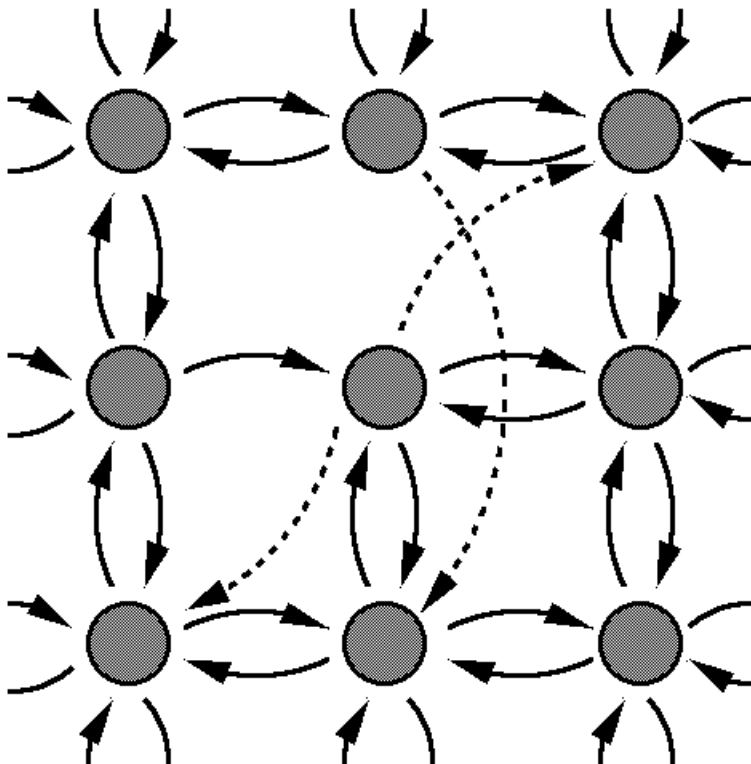
Figure 1: Sketch of a *directed* small-world networks constructed from a square regular lattice in $d = 2$. Figure gently yielded by Juan M. Lopez of the paper of Sánchez et al. [19] .

see Fig. 1, we start from a two-dimensional square lattice consisting of sites linked to their four nearest neighbors by both outgoing and incoming links. Then, with probability p, we reconnect nearest-neighbor outgoing links to a different site chosen at random. After repeating this process for every link, we are left with a network with a density p of SW *directed* links. Therefore, with this procedure every site will have exactly four outgoing links and a varying (random) number of incoming links. The control parameter q plays the role of the temperature in equilibrium systems and measures the probability of aligning antiparallel to the majority of neighbours.

To study the critical behavior of the model we define the variable $m = \sum_{i=1}^{N} \sigma_i / N$. In particular, we were interested in the magnetisation, suscepti-

bility and the reduced fourth-order cumulant:

$$M(q) = [< |m| >]_{av}, \tag{2}$$

$$\chi(q) = N[< m^2 > - < |m| >^2]_{av}, \tag{3}$$

$$U(q) = \left[1 - \frac{< m^4 >}{3 < |m| >^2}\right]_{av}, \tag{4}$$

where $< ... >$ stands for a thermodynamics average and $[...]_{av}$ square brackets for a averages over the 20 realizations.

These quantities are functions of the noise parameter q and obey the finite-size scaling relations

$$M = L^{-\beta/\nu} f_m(x)[1 + ...], \tag{5}$$

$$\chi = L^{\gamma/\nu} f_\chi(x)[1 + ...], \tag{6}$$

$$\frac{dU}{dq} = L^{1/\nu} f_U(x)[1 + ...], \tag{7}$$

where ν, β, and γ are the usual critical exponents, $f_i(x)$ are the finite size scaling functions with

$$x = (q - q_c)L^{1/\nu} \tag{8}$$

being the scaling variable, and the brackets $[1 + ...]$ indicate corretions-to-scaling terms. Therefore, from the size dependence of M and χ we obtained the exponents β/ν and γ/ν, respectively. The maximum value of susceptibility also scales as $L^{\gamma/\nu}$. Moreover, the value of q for which χ has a maximum, $q_c^{\chi max} = q_c(L)$, is expected to scale with the system size as

$$q_c(L) = q_c + bL^{-1/\nu}, \tag{9}$$

were the constant b is close to unity. Therefore, the relations (7) and (9) are used to determine the exponent $1/\nu$.

We have performed Monte Carlo simulation on *directed* SW network with various values of probability p. For a given p, we used systems of size $L = 8$, 16, 32, 64, and 128. We waited 10000 Monte Carlo steps (MCS) to make the system reach the steady state, and the time averages were estimated from the next 10000 MCS. In our simulations, one MCS is accomplished after all the N spins are updated. For all sets of parameters, we have generated 20 distinct networks, and have simulated 20 independent runs for each distinct network.

Results and Discussion

In Fig. 2 we show the dependence of the magnetisation M and the susceptiblity χ on the noise parameter, obtained from simulations on *directed* SW network with $L = 128 \times 128$ sites and several values of probability p. In part (a) each curve for M, for a given value of L and p, suggests that there is a phase transition from an ordered state to a disordered state. The phase transition occurs at a value of the critical noise parameter q_c, which is an increasing function the probability p of the *directed* SW network. In part (b) we show the corresponding behavior of the susceptibility χ, the value of q where χ has a maximum is here identified as q_c. In Fig. 3 we plot Binder's fourth-order cumulant for different values of L and two different values of p. The critical noise parameter q_c, for a given value of p, is estimated as the point where the curves for different system sizes L intercept each other. In Fig 4 the phase diagram is shown as a function of the critical noise parameter q_c on probability p obtained from the data of Fig. 3.

The phase diagram of the majority-vote model on *directed* SW network shows that for a given network (fixed p) the system becomes ordered for $q < q_c$, whereas it has zero magnetisation for $q \geq q_c$. We notice that the increase of q_c as a function the p is slower that the one in [21]. In Figs. 5 and 6 we plot the dependence of the magnetisation and susceptibility, respectively, at $q = q_c$ versus the system size L. The slopes of curves correspond to the exponent ratio β/ν and γ/ν of according to Eq. (5) and (6), respectively. The results show that the exponent ratio β/ν and γ/ν at q_c are independent of p (along with errors), see Table I.

In Fig. 7 we display the scalings for susceptibility at $q = q_c(L)$ (square), $\chi(q_c(L))$, and for its maximum amplitude, χ_L^{max}, and the scalings for susceptibility at the $q = q_c$ obtained from Binder's cumulant, $\chi(q_c)$ (circle), versus L for probability $p = 0.5$. The exponents ratio γ/ν are obtained from the slopes of the straight lines. For almost all the values of p, the exponents γ/ν of the two estimates agree (along with errors). We also observe that an increased p does not mean a tendency to increase or decrease the exponent ratio γ/ν, see Table I. Therefore we can use the Eq. (9) , for fixed p, to obtain the critical exponent $1/\nu$, see Fig. 8.

To improve our results obtained above we start with all spins up, a number of spins equal to $N = 640000$, and time up 2,000,000 (in units of Monte Carlo steps per spins). Then we vary the noise parameter q and at each q study the time dependence for 9 samples. We determine the time τ after which the magnetisation has flipped its sign for fisrt time, and then take the

median values of our nine samples. So we get different values τ_1 for different noise parameters q. In Fig. 9 show that the decay time goes to infinity at some q positive values this behavior sure that there is a phase transition for Majority-vote on directed SW network.

The Table I summarizes the values of q_c, the exponents β/ν, γ/ν, and $1/\nu$. J. M. Oliveira [13] showed that the majority-vote model defined on a regular lattice has critical exponents that fall into the same class of universality as the corresponding equilibrium Ising model. Campos et al [17] investigated the critical behavior of the majority-vote on small-world networks by rewiring the two-dimensional square lattice, Pereira et al [21] studied this model on Erdös-Rényi's random graphs, and Lima et al [18] also studied this model on Voronoy-Delaunay lattice and Lima on *directed* Barabási-Albert network [15]. The results obtained these authors show that the critical exponents of majority-vote model belong to different universality classes.

p	q_c	β/ν	γ/ν^{q_c}	$\gamma/\nu^{q_c(L)}$	$1/\nu$
0.1	0.122(3)	0.423(17)	1.178(13)	1.214(39)	0.837(223)
0.3	0.149(3)	0.419(21)	1.148(5)	1.152(28)	1.059(208)
0.5	0.160(2)	0.441(12)	1.116(5)	1.120(25)	1.010(52)
0.8	0.164(2)	0.436(9)	1.149(5)	1.117(23)	1.248(158)
1.0	0.165(2)	0.415(18)	1.139(8)	1.122(25)	1.032(81)

Table 1: The critical noise q_c, and the critical exponents , for *directed* SW network with probability p. Error bars are statistical only.

Finally, we remark that our MC results obtained on *directed* SW network for majority-vote model show that critical exponents are different from the results of [13] for regular lattice, of Pereira et al [21] for Erdös-Rényi's random graphs, Lima [15] and Campos *et al.* [17] for on *undirected* SW network.

Conclusion

In conclusion, we have presented a very simple nonequilibrium model on *directed* SW network [19]. In these networks, the majority-vote model presents a second-order phase transition which occurs with probability $p \geq 0$. The exponents obtained are different from the other models [13, 14, 15, 16, 17, 21] suggesting that these exponents belong to another class of universality. However, the exponents in the critical point q_c, β/ν, γ/ν and $1/\nu$ when p

grows no increase and do not decrease and are independents of a growing p for $p > 0$.

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References

[1] M.A. Sumour and M.M. Shabat, Int. J. Mod. Phys. C 16, 585 (2005) and cond-mat/0411055 at www.arXiv.org.

[2] M.A. Sumour, M.M. Shabat and D. Stauffer, Islamic University Journal (Gaza) 14, 209 (2006).

[3] R. Albert and A.L. Barabási, Rev. Mod. Phys. 74, 47 (2002).

[4] A. Aleksiejuk, J.A. Hołyst and D. Stauffer, Physica A 310, 269 (2002).

[5] J.O. Indekeu, Physica A 333, 461 (2004).

[6] G. Bianconi, Phys. Lett. A 303, 166 (2002).

[7] F.W.S. Lima and D. Stauffer, Physica A 359, 423 (2006).

[8] J.S. Wang and R. H. Swendsen, Physica A 167, 565 (1990).

[9] G. Grinstein, C. Jayapralash, and Yu He, Phys. Rev. Lett. 55, 2527 (1985).

[10] C.H. Bennett, Phys. Rev. Lett. 55, 657 (1985).

[11] J.S. Wang and J.L. Lebowitz, J. Stat. Phys. 51, 893 (1988).

[12] M.C. Marques, Phys. Lett. A 145, 379 (1990).

[13] M.J. Oliveira, J. Stat. Phys. 66, 273 (1992).

[14] F.W.S. Lima, Int. J. Mod. Phys. C 17, 9, 1273 (2006).

[15] F.W.S. Lima, Int. J. Mod. Phys. C 17, 9, 1257 (2006).

[16] F.W.S. Lima, Commun. Comput. Phys. 2, 358 (2007).

[17] P.R. Campos, V.M. Oliveira, and F.G.B.Moreira, Phys. Rev. E 67, 026104 (2003).

[18] F.W.S. Lima, U.L. Fulco, and R.N. Costa Filho, Phys. Rev. E 71, 036105 (2005).

[19] Alejandro D. Sánchez, Juan M. Lopes, and Miguel A. Rodriguez, Phys. Rev. Lett. 88, 048701 (2002)

[20] J.J.F. Mendes and M. A. Santos, Phys. Rev. E 57, 108 (1998)

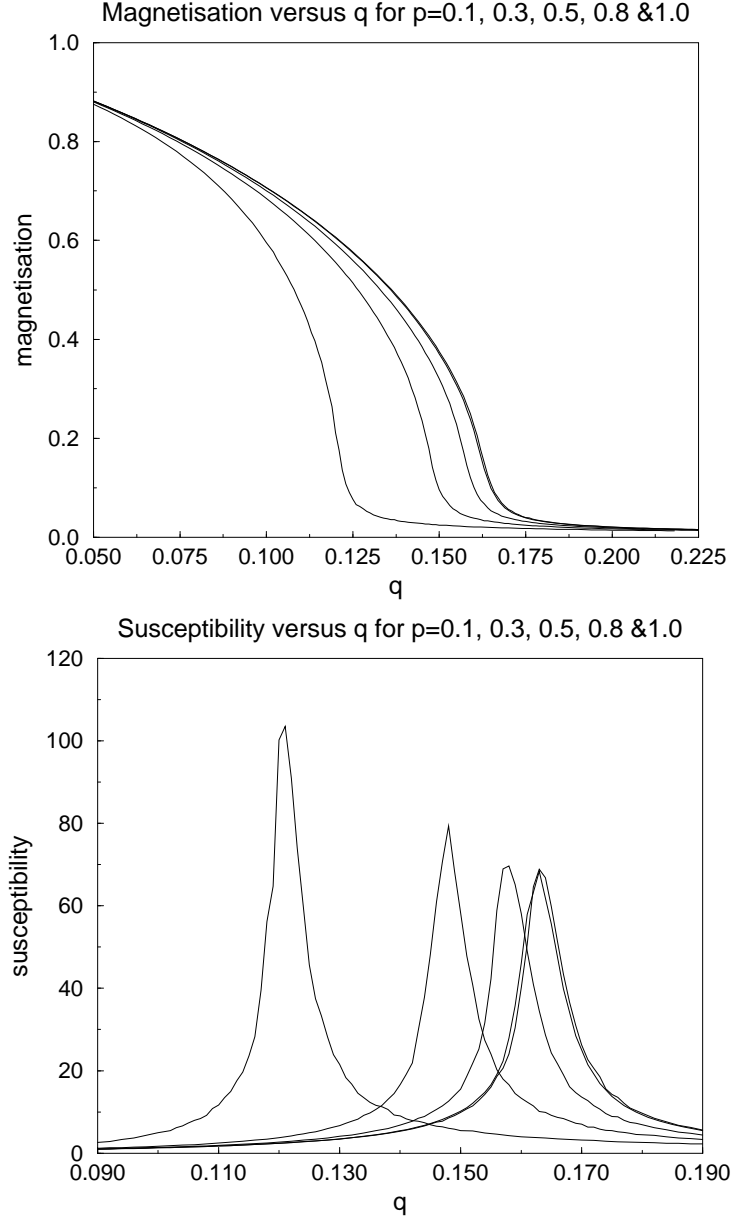[21] Luiz F. C. Pereira and F.G. Brady Moreira, Phys. Rev. E 71, 016123 (2005).

Figure 2: Magnetisation and susceptibility as a function of the noise parameter q, for $N = 16384$ sites. From left to right we have $p = 0.1$, 0.3, 0.5, 0.8, and 1.0 .
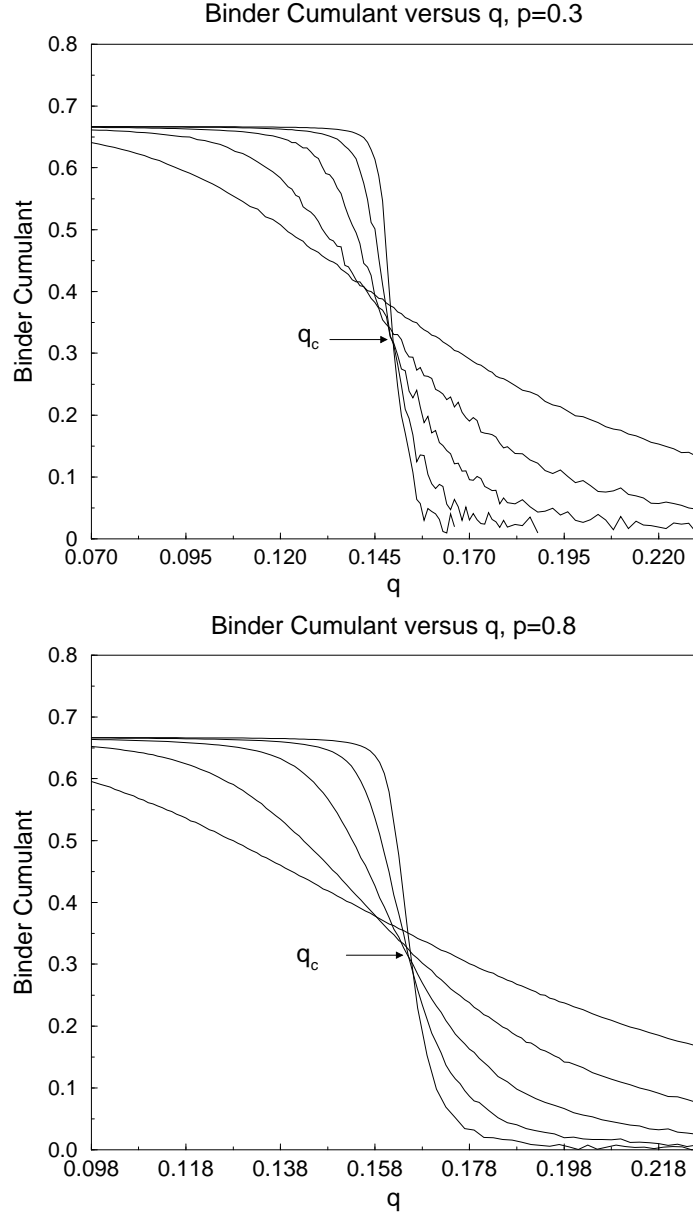
Figure 3: Binder's fourth-order cumulant as a function of q. We have $p = 0.3$ and $p = 0.8$ for $L = 8$, 16, 32, 64 and 128.
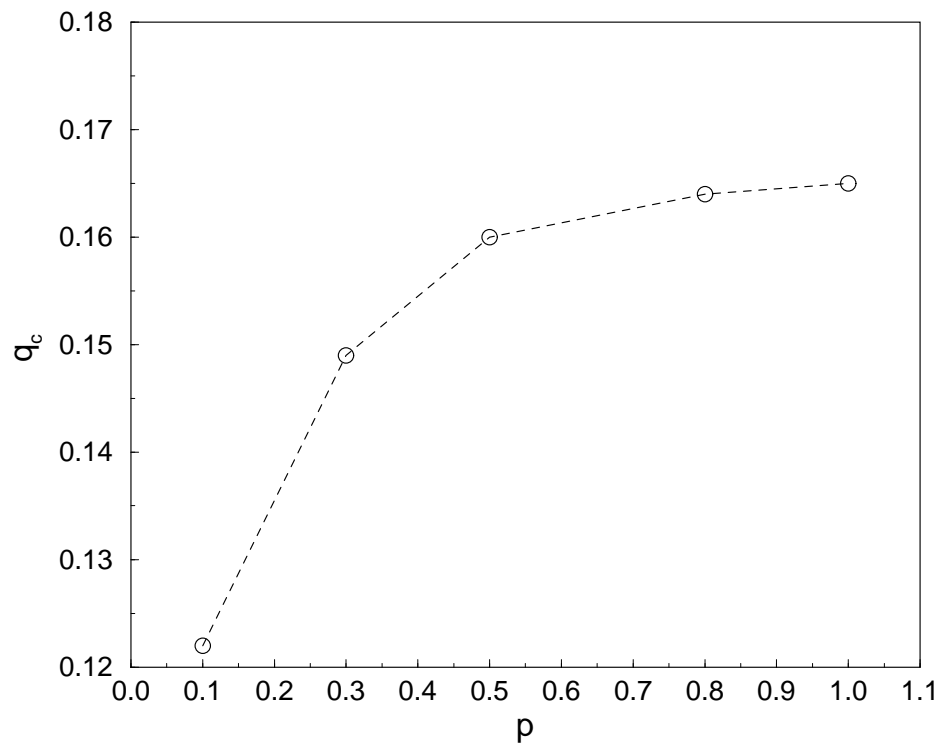
Figure 4: The phase diagram, showing the dependence of critical noise parameter q_c on probability p.
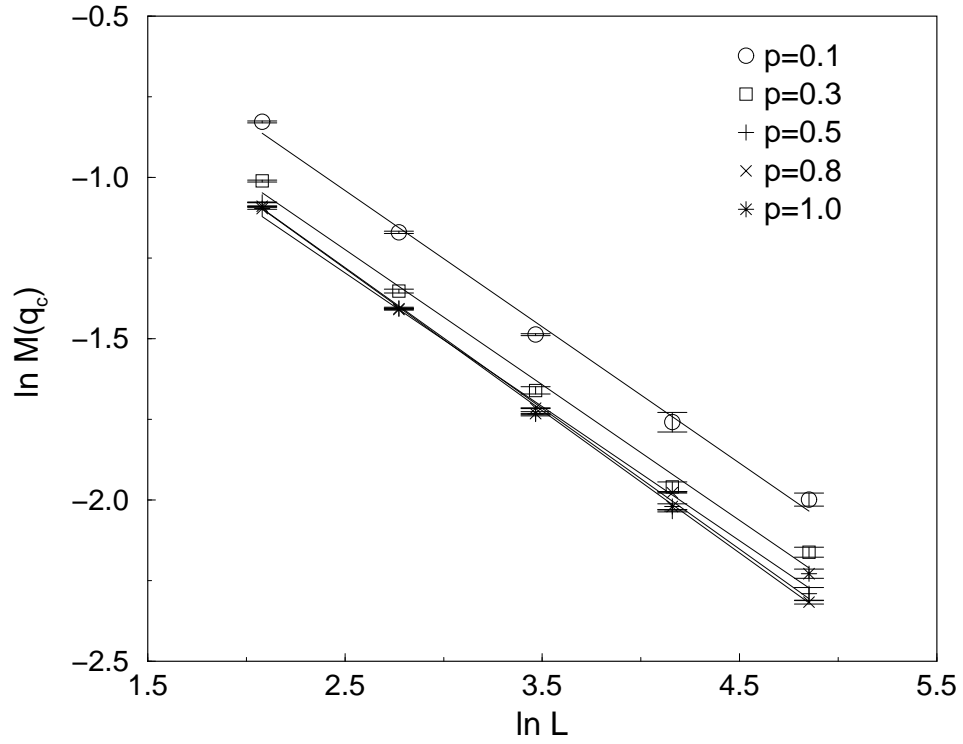
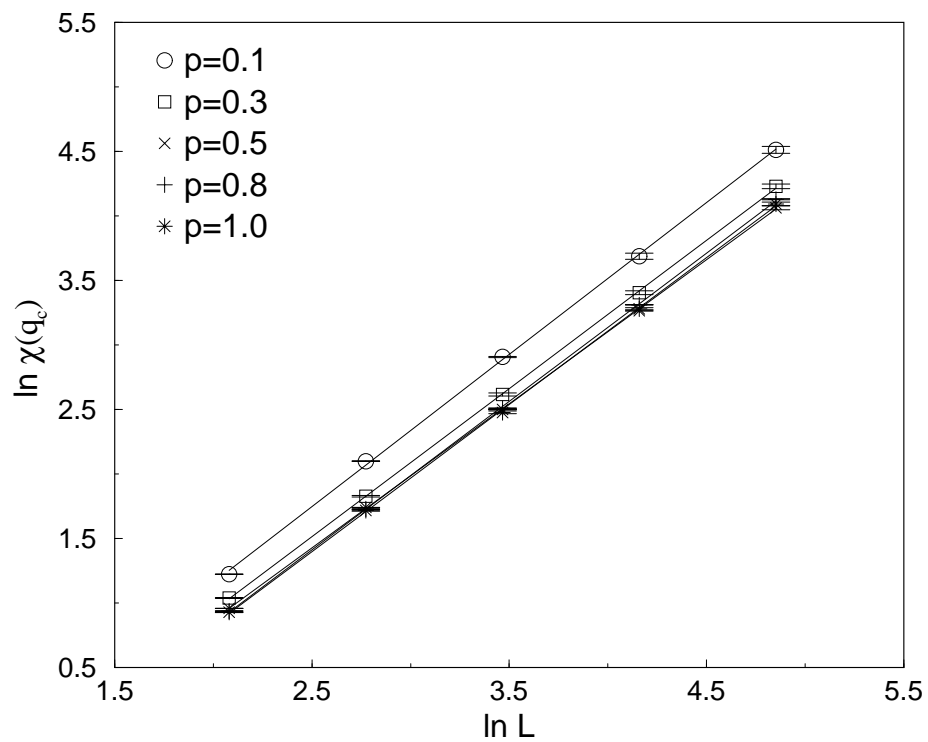Figure 5: $\ln M(q_c)$ versus $\ln L$. From top to bottom, $p = 0.1$, 0.3, 0.5, 0.8, and 1.0.

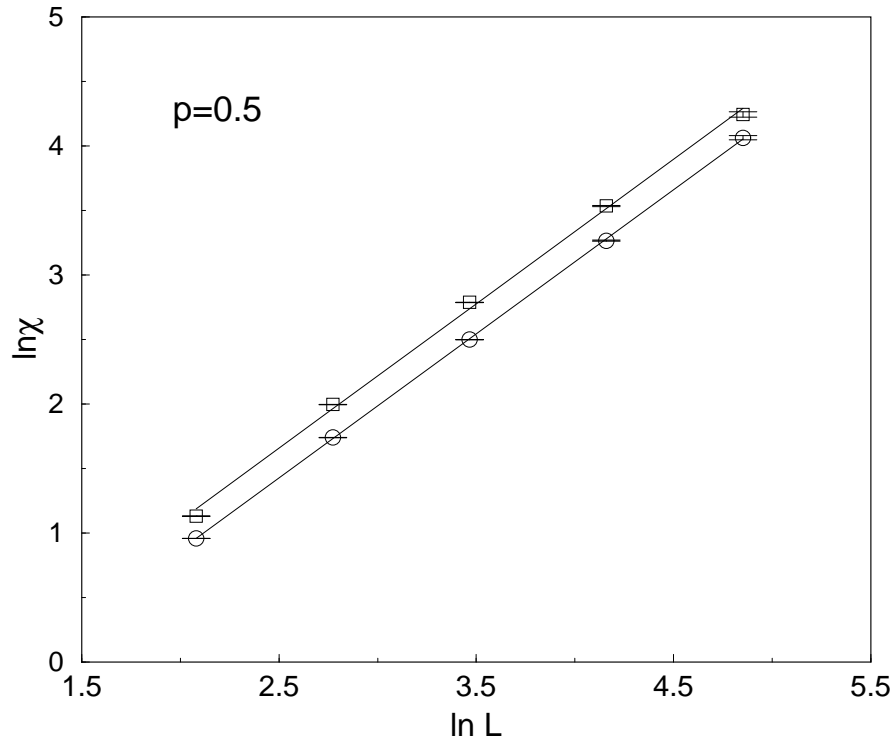Figure 6: $\ln \chi(q_c)$ versus $\ln L$. From top to bottom $p = 0.1$, 0.3, 0.5, 0.8, and 1.0.

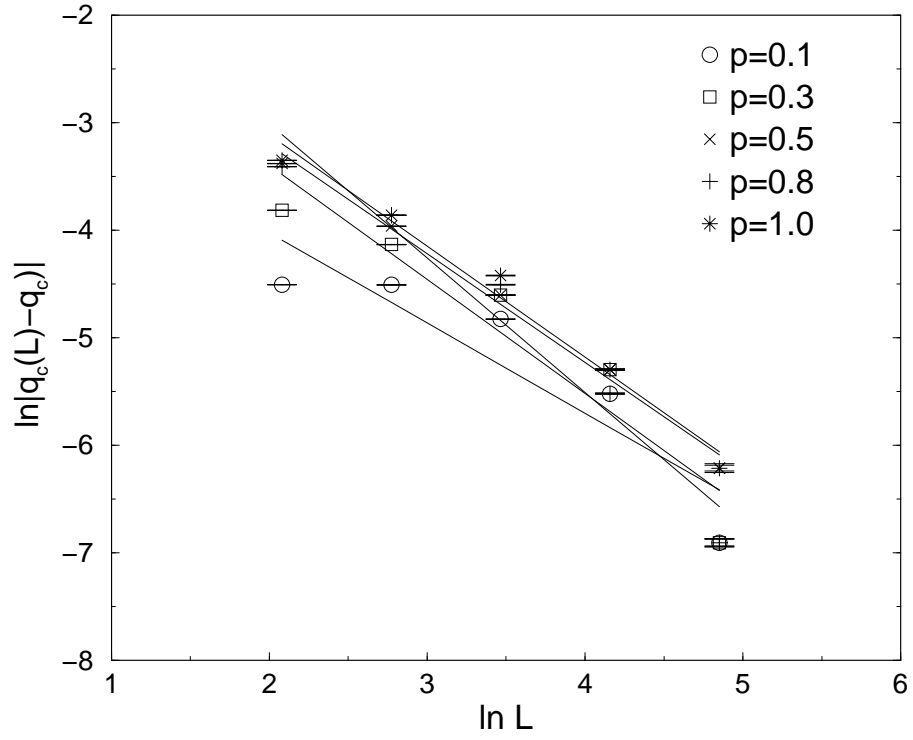Figure 7: Plot of $\ln \chi^{max}(L)$ (square) and $\ln\chi(q_c)$ (circle) versus $\ln L$ for connectivity $p = 0.5$.

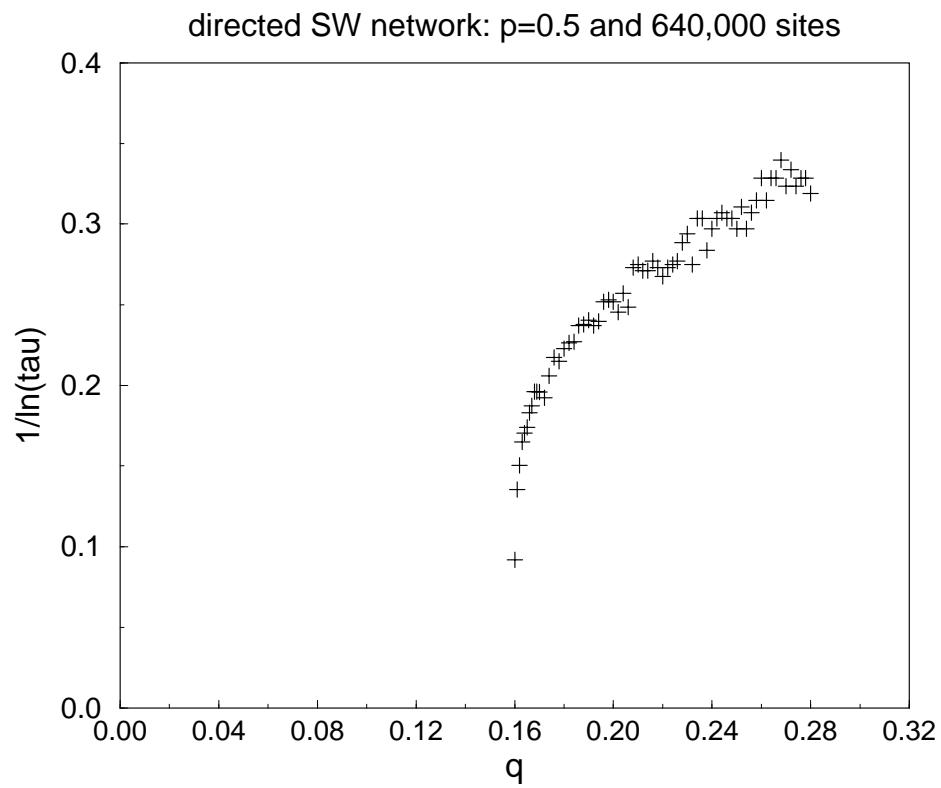Figure 8: Plot of $\ln|q_c(L) - q_c|$ versus $\ln L$. From bottom to top $p = 0.1$, 0.3, 0.5, 0.8, and 1.0.

directed SW network: p=0.5 and 640,000 sites

Figure 9: Plot of 1/ln(tau) versus q for p=0.5.